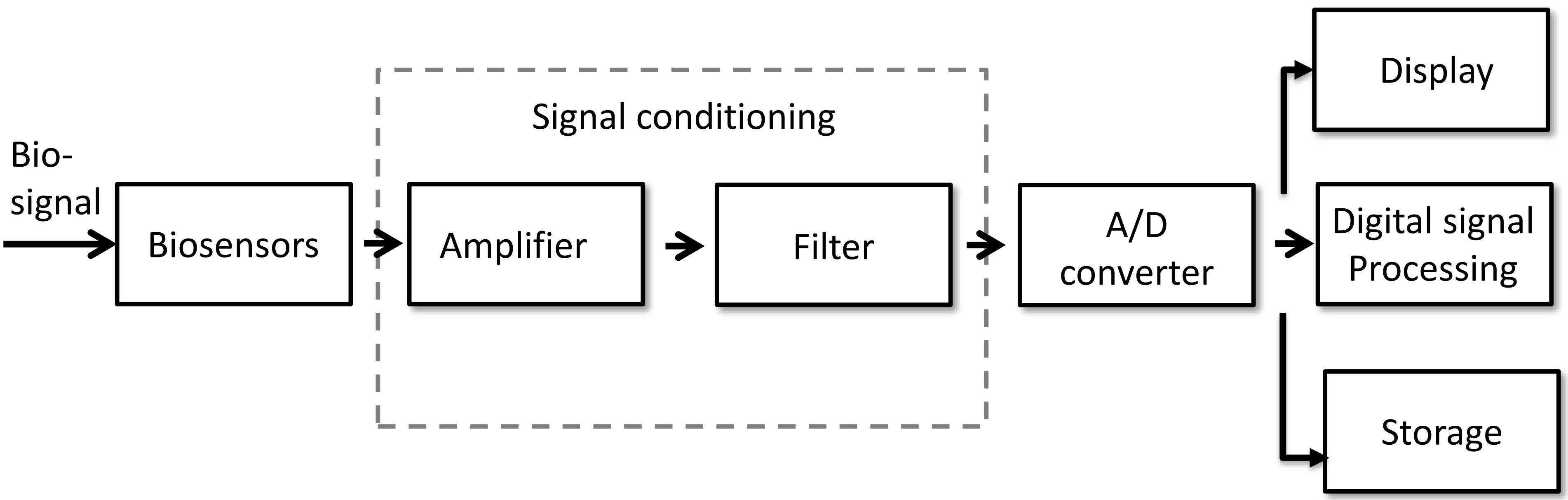
Linear circuit analysis

COLUMBIA ENGINEERING
The Fu Foundation School of Engineering and Applied Science

Biomedical instrumentation



Linear system

- Response obeys the principle of superposition.

$$f(\alpha \cdot x) = \alpha \cdot f(x)$$

$$f(x_1 + x_2) = f(x_1) + f(x_2)$$

$$f(\alpha \cdot x_1 + \beta \cdot x_2) = \alpha \cdot f(x_1) + \beta \cdot f(x_2)$$

$$\alpha, \beta = const.$$

- Response can be expressed as the convolution of the input with the unit impulse response of the system.

Linear system: examples

$$f(x) = 5 \cdot x$$

$$f(x) = x^2$$

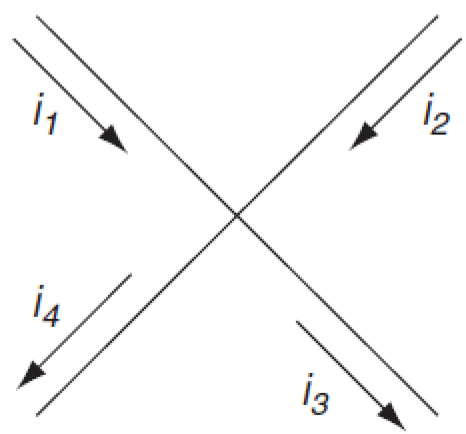
$$f(x) = Sin(x)$$

$$f(x) = 2 + 5 \cdot x$$

$$f(x) = \int_0^s x(t)dt$$

Kirchhoff's Current Law (KCL)

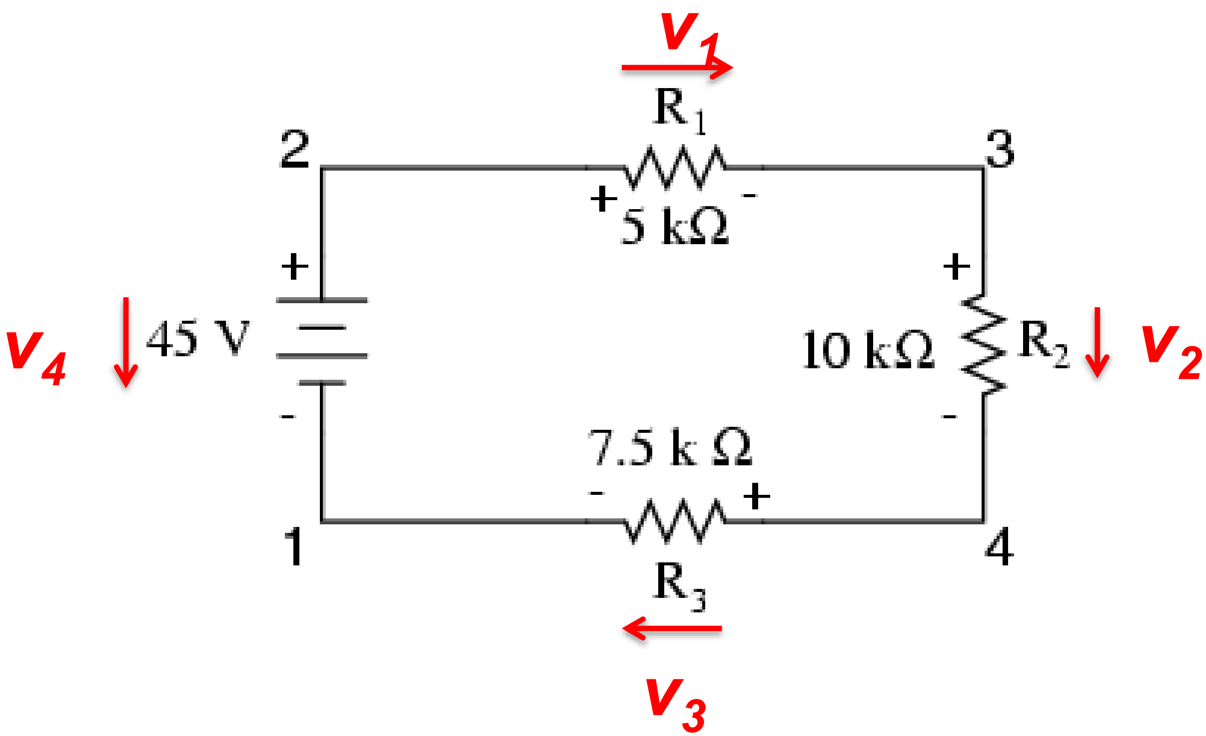
- No current is lost as it flows around the circuit because charge must be conserved.
- Since charge must be conserved, the sum of the currents at any node (i.e. a point at which two or more circuit elements have a common connection) must equal zero, so no net charge accumulates.



$-i_1 - i_2 + i_4 + i_3 = 0$ or $i_1 + i_2 - i_4 - i_3 = 0$

Kirchhoff’s voltage law (KVL)

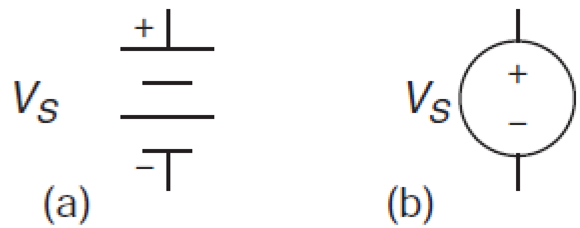
- The sum of all voltages in a closed path is zero



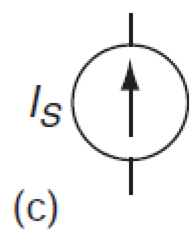
$$v_1 + v_2 + v_3 - v_4 = 0$$

Basic components

- **Sources**: terminal devices that provide energy to a circuit.
  - Ideal voltage source: a device that generates a prescribed voltage at its terminals, regardless of the current flow

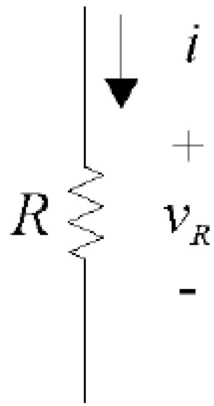


  - Ideal current source: delivers a prescribed current to the attached circuit



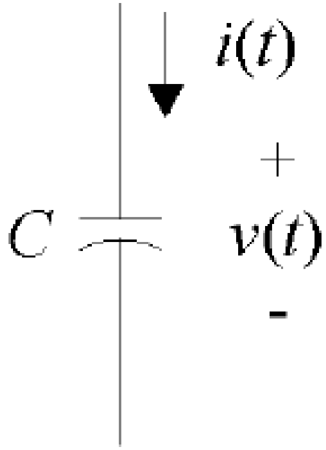
Basic components

- **Resistor**



$$i(t) = \frac{V(t)}{R}$$

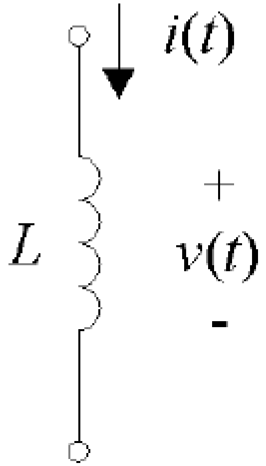
- **Capacitor**



$$i(t) = C \cdot \frac{dV(t)}{dt}$$

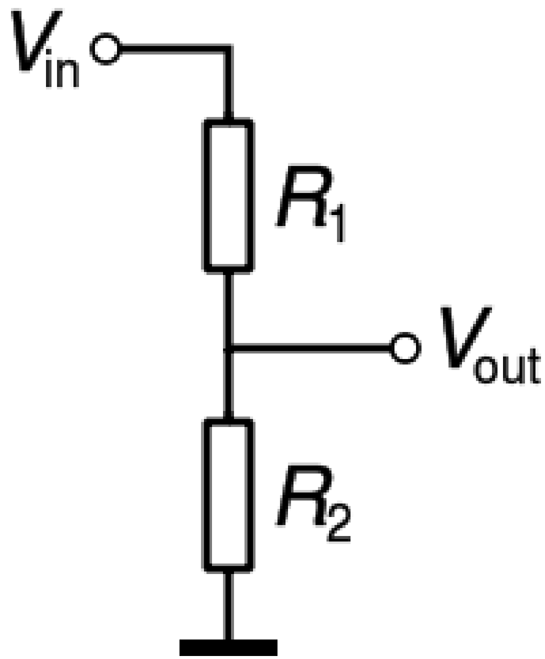
Basic components

- **Inductor**



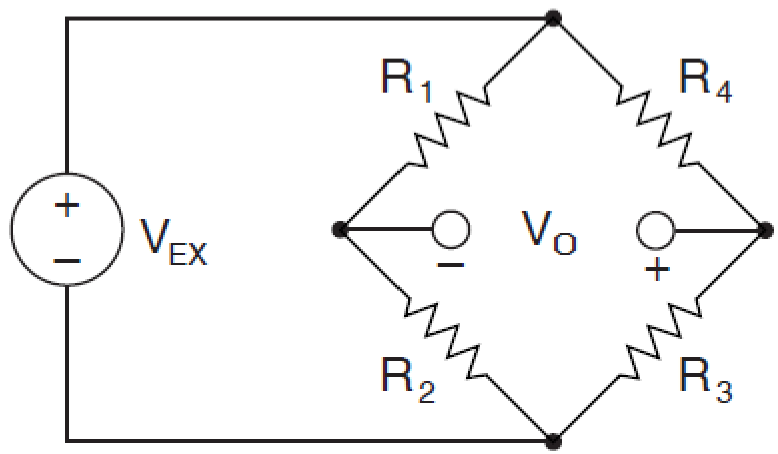
$$V(t) = L \cdot \frac{di(t)}{dt}$$

Voltage divider



$$V_{out} = \frac{R_2}{R_1 + R_2} \cdot V_{in}$$

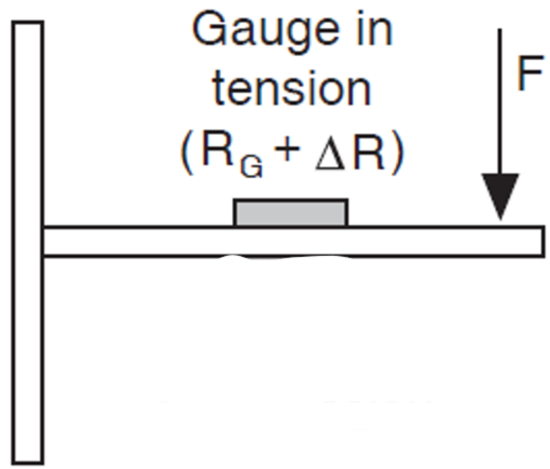
Wheatstone Bridge

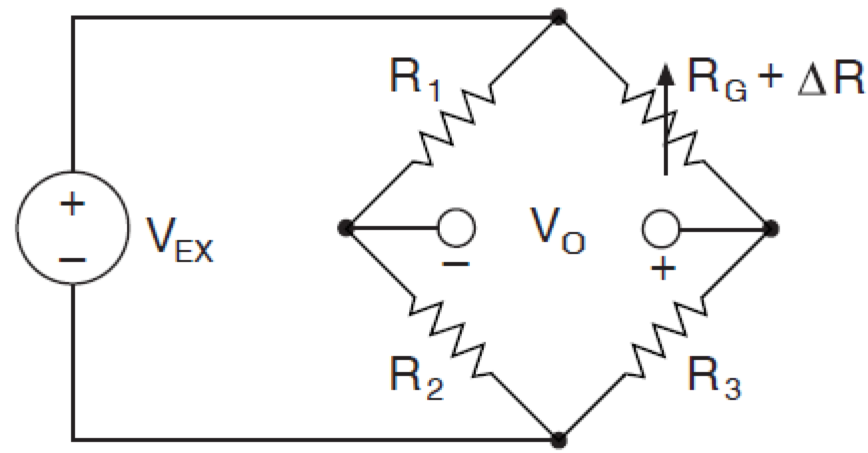


$$V_O = \left(\frac{R_3}{R_3 + R_4} - \frac{R_2}{R_1 + R_2}\right)V_{EX}$$

- If $R_4/R_3 = R_1/R_2$, the Vo =0, which means the bridge is balanced.

Quarter Wheatstone Bridge





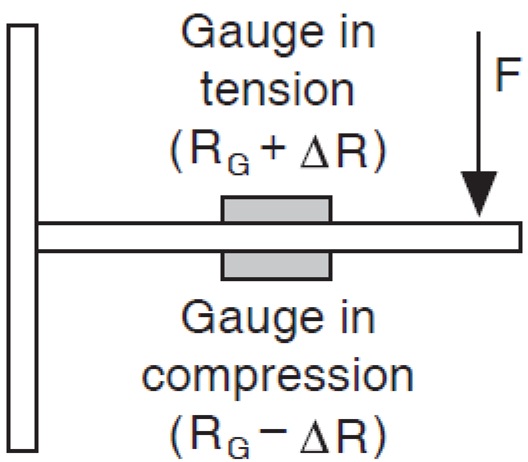
- If $R_1 = R_2$, and $R_3 = R_G$,

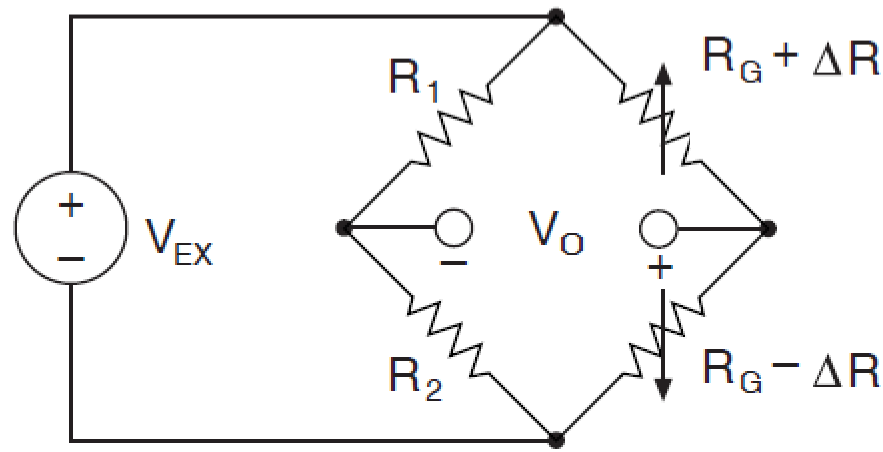
$$V_0 = -\frac{G\cdot\varepsilon}{4}\frac{1}{1+G\cdot\varepsilon/2}\cdot V_{EX}$$

$$\approx -\frac{G\cdot\varepsilon}{4}\cdot V_{EX}$$

Where G is gage factor

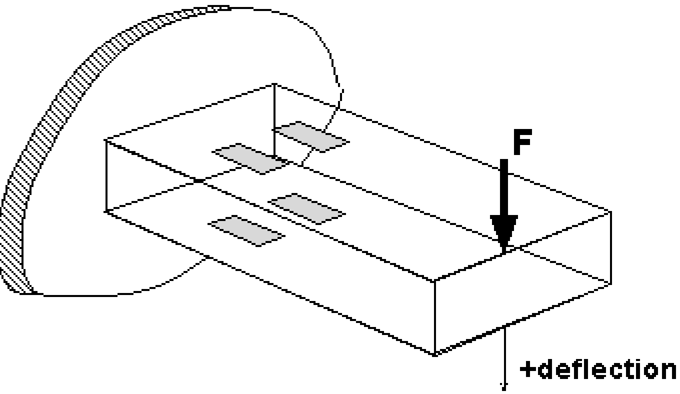
Half Wheatstone Bridge

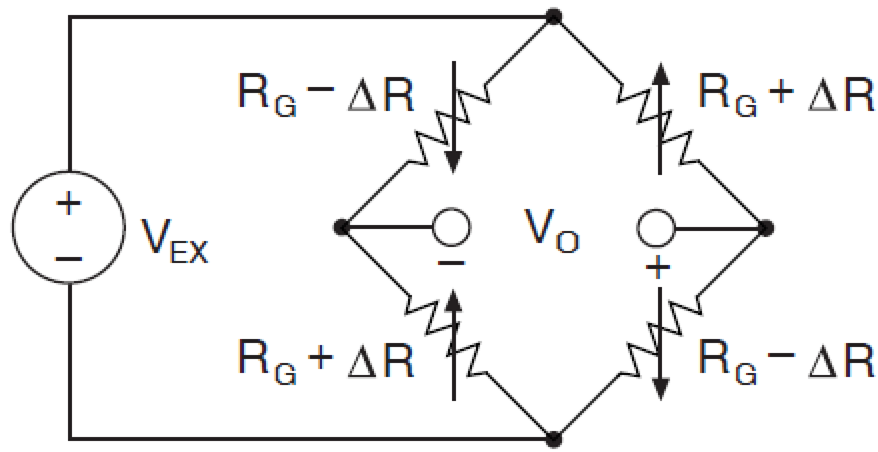




$$V_0 = -\frac{G \cdot \varepsilon}{2} V_{EX}$$

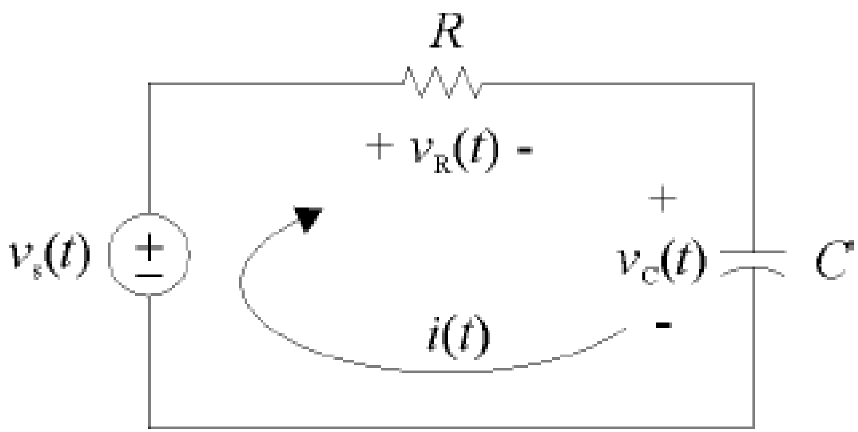
Full Wheatstone Bridge





$$V_0 = -G \cdot \varepsilon \cdot V_{EX}$$

RC circuit



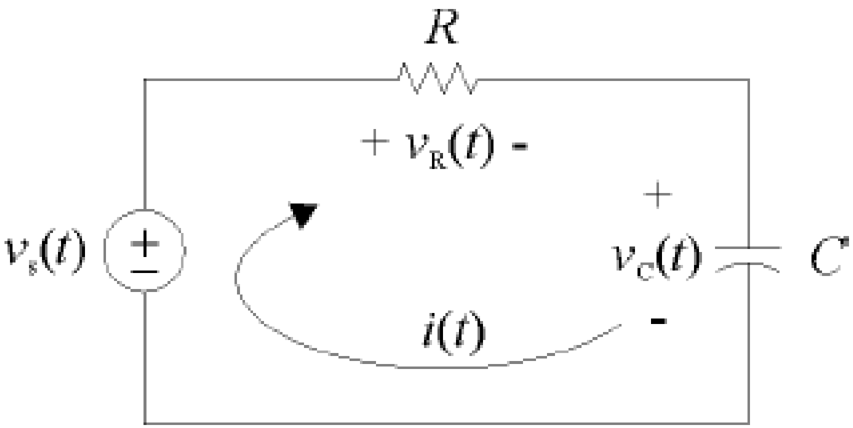
- $v_s(t)$ is a step function input.
- What is the voltage across the capacitor, $v_c(t)$?

$$v_s(t) = RC\frac{dv_c(t)}{dt} + v_c(t)$$

$$RC\frac{dv_c(t)}{dt} = v_s(t) - v_c(t) = 1 - v_c(t)$$

$$v_c(t) = 1 - e^{-\frac{t}{RC}}$$

RC circuit



- What if $v_s(t)$ is a sinusoidal signal?

$$v_s(t) = \sin(\varpi \cdot t)$$

$$RC\frac{dv_c(t)}{dt} = v_s(t) - v_c(t)$$

$$v_c(t) = ?$$

Laplace transform

- The Laplace transform converts integral and differential equations into algebraic equations.
- Allow us to analyze complicated systems with integrators and differentiators.
- The Laplace transform of a function, *f(t)*, is defined as:

$$L[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}dt$$

Laplace transform

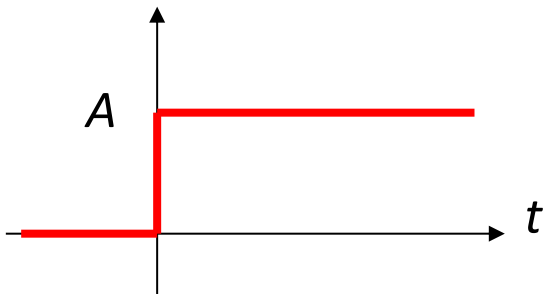
- The Inverse Laplace Transform is defined by

$$L^{-1}[F(s)] = f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{ts} ds$$

Laplace transform of basic functions

- Laplace Transform of the unit step function.

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$



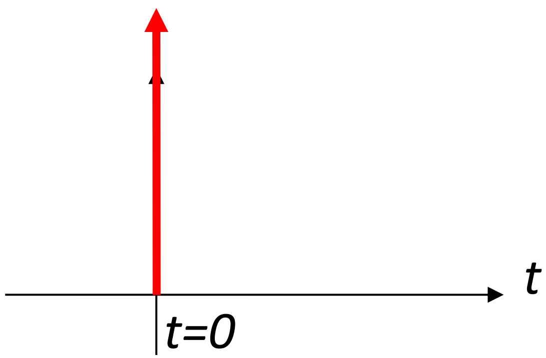
$$L[u(t)] = \int_0^\infty 1e^{-st}\,dt = \frac{-1}{s}e^{-st}\Big|_0^\infty$$

$$L[u(t)] = \frac{1}{s}$$

Laplace transform of basic functions

- Laplace Transform of the delta function.

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$



$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

$$\int_{-\infty}^{\infty} \delta(t - t_0) f(t)\,dt = f(t_0)$$

Laplace transform of delta functions

$$L[\delta(t)] = \int_0^\infty \delta(t) e^{-st} dt = 1$$

$$L[\delta(t-\alpha)] = \int_0^\infty \delta(t-\alpha) e^{-st} dt = e^{-\alpha s}$$

Laplace transform pairs

- As long as f(t) doesn't grow faster than an exponential function, for each f(t) in time domain there is a **unique** F(s) in Laplace domain (frequency domain) and for each F(s) there is a **unique** f(t). f(t) and F(s) are therefore transform pairs.

$$f(t) \quad \Leftrightarrow \quad F(s)$$

- Check Laplace transform table for most popularly used Laplace transform pairs.

Laplace transform properties

- Time Differentiation:

$$L[\frac{df(t)}{dt}] = sF(s) - f(0)$$

Laplace transform properties

$$L\left[\frac{d^2 f(t)}{dt^2}\right] = s^2 F(s) - sf(0) - f'(0)$$

$$L\left[\frac{d^3 f(t)}{dt^3}\right] = s^3 F(s) - s^2 f(0) - sf'(0) - f''(0)$$

$$general\ case$$

$$L\left[\frac{d^n f(t)}{dt^n}\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0)$$

$$-\ldots - f^{(n-1)}(0)$$

Laplace transform properties

- Time Integration:

$$L\left[\int_0^t f(x)dx\right] = \int_0^\infty \left[\int_0^t f(x)dx\right]e^{-st}dt$$

$$= \frac{1}{s}\int_0^\infty f(t)e^{-st}dt$$

$$= \frac{1}{s}F(s)$$

Laplace transform properties

- If

$$f(t) \quad \Leftrightarrow \quad F(s)$$

$$g(t) \quad \Leftrightarrow \quad G(s)$$

- Then:

$$f(t)+g(t) \quad \Leftrightarrow \quad F(s)+G(s)$$

$$f(t)*g(t) \quad \Leftrightarrow \quad F(s)\cdot G(s)$$

Laplace transform to solve ODEs

- Find y(t)

$$2\dot{y}(t) + y(t) = 1 \quad \longrightarrow \quad y(t) = 1 - e^{-\frac{t}{2}}$$

Laplace domain: $2sY(s) + Y(s) = \dfrac{1}{s}$ $\quad L[1] = \dfrac{1}{s}$!!!

$$Y(s) = \frac{1}{s\cdot(2s+1)} = \frac{1}{s} - \frac{2}{2s+1}$$

$$\downarrow$$

$$y(t) = 1 - e^{-\frac{t}{2}}$$